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so(3, 1) versus sp(4, ℝ) as dynamical potential algebra of the symmetrical Pöschl–Teller potentials

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Abstract. The results of a previous paper, concerned with the first family of two-parameter Pöschl-Teller potentials, are used to obtain potential and dynamical potential algebras for the subfamily of one-parameter symmetrical Pöschl-Teller potentials. They are respectively identified with so(3) and so(3, 1) algebras, and shown to be subalgebras of the so(4) potential and sl(4, ℝ) dynamical potential algebras of the two-parameter potentials. All the Hamiltonian eigenstates, corresponding to the subfamily of potentials with quantised potential strengths μ differing by an integer, are proved to belong to a single degenerate continuous unitary irreducible representation of so(3, 1), which may be labelled by generalised Young pattern labels $[\omega 0]$, where $\omega = -1 + \frac{1}{4}\mathrm{i}\eta$, and η is some real parameter. An sp(4, ℝ) ≃ so(3, 2) algebra is also constructed and shown to provide an alternative choice for the dynamical potential algebra of the symmetrical potentials. All the above-mentioned eigenstates belong to a single sp(4, ℝ) unitary irreducible representation of the positive discrete series, which may be labelled by its lowest weight $\langle\frac{1}{2}\frac{1}{2}\rangle$. Such an alternative choice for the dynamical potential algebra, however, does not lead to a unified treatment of the one- and two-parameter potentials as does the first one.

1. Introduction

In the preceding paper (Quesne 1988, hereafter referred to as I), we extended the so(4) potential algebra of the first family of two-parameter Pöschl-Teller potentials (Barut *et al* 1987) to an sl(4, ℝ) dynamical potential algebra. We based this construction on a relation between the solutions of the first Pöschl-Teller equation and the Wigner rotation matrices, and showed that all the eigenstates, corresponding to the family of potentials with quantised potential strengths (m', m) differing by integers, belong to the carrier space of a single sl(4, ℝ) ladder unitary irreducible representation (irrep), $\mathfrak{D}^{\mathrm{ladd}}(0, 0; \eta)$ or $\mathfrak{D}^{\mathrm{ladd}}(\frac{1}{2}, \frac{1}{2}; \eta)$, according as m' and m are integral or half-integral, respectively.

In the present paper, we show that the results of I can be easily specialised to the subfamily of one-parameter symmetrical Pöschl-Teller potentials. For the latter, the solutions of the Schrödinger equation can be simply related to the spherical harmonics. We also prove that potential and dynamical potential algebras for this subfamily are provided by so(3) and so(3, 1) subalgebras of so(4) and sl(4, ℝ), respectively.

We then contrast this approach with another one, leading to an sp(4, ℝ) ≃ so(3, 2) dynamical potential algebra. The latter is similar to that used by Alhassid *et al* (1983,

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1986), dealing with a corresponding subset of the second family of Pöschl–Teller potentials, and is also related to a recent algebraic study of the spherical harmonics (Humi 1987). We emphasise and propose a solution to a problem, previously left unnoted, arising in the construction of sp(4, $\mathbb{R}$).

2. The one-parameter Pöschl–Teller equation

By setting $\kappa = \lambda$ in the two-parameter Pöschl–Teller potential (Pöschl and Teller 1933)

$$V = \frac{\hbar^2 a^2}{2M}\left(\frac{\kappa(\kappa-1)}{\sin^2(ax)} + \frac{\lambda(\lambda-1)}{\cos^2(ax)}\right) \qquad \kappa, \lambda > 1 \tag{2.1}$$

where a defines the range of x ($x \in [0, \pi/2a]$), we obtain a one-parameter potential, symmetrical around $\pi/4a$. The corresponding Schrödinger equation can be written as

$$\left[-\frac{\hbar^2}{2M}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{2\hbar^2 a^2}{M}\frac{\kappa(\kappa-1)}{\sin^2(2ax)} - E_n\right]\psi_n(x) = 0 \qquad \kappa > 1 \tag{2.2}$$

where $n \in \mathbb{N}$ labels the eigenvalues E_n and the wavefunctions $\psi_n(x)$. In terms of the alternative parametrisation (m', m) used in I, the condition $\kappa = \lambda > 1$ leads to the relations

$$m = 0 \qquad \kappa = \mu + \tfrac{1}{2} \qquad \mu > \tfrac{1}{2} \tag{2.3}$$

where μ is substituted for m'. In accordance with (2.3), the wavefunctions will henceforth be denoted by $\psi_n^{(\mu)}(x)$.

As in I, the solutions of (2.2) can be obtained by performing the change of variable

$$x = (\pi - \theta)/2a \qquad \theta \in [0, \pi] \tag{2.4}$$

where θ is now used instead of β. The eigenvalues are given by

$$E_n = 2\hbar^2 a^2 \Lambda_n / M \qquad \Lambda_n = (\mu + n + \tfrac{1}{2})^2 = (\kappa + n)^2 \qquad n \in \mathbb{N}. \tag{2.5}$$

Provided that

$$l = \mu + n \qquad n \in \mathbb{N} \tag{2.6}$$

where l and μ are restricted to positive integral values, the wavefunctions can be expressed as

$$\begin{aligned}\psi_n^{(\mu)}(x) &= [(2l+1)a \sin\theta]^{1/2} d^l_{\mu 0}(\theta) \\ &= (-1)^\mu [(l+\mu)!]^{-1/2}[(2l+1)(l-\mu)!\, a \sin\theta]^{1/2} P_l^\mu(\cos\theta)\end{aligned} \tag{2.7}$$

where $d^l_{\mu 0}(\theta)$ is a Wigner rotation matrix element, and $P_l^\mu(\cos\theta)$ an associated Legendre function of the first kind (Biedenharn and Louck 1981).

By introducing an additional dependence on one auxiliary angular variable $\varphi \in [0, 2\pi)$, which plays the same role as the variable α in I, the functions (2.7) are transformed into the extended wavefunctions

$$\begin{aligned}\Psi_n^{(\mu)}(x, \varphi) &= (2\pi)^{-1/2} \exp(\mathrm{i}\mu\varphi)\psi_n^{(\mu)}(x, \varphi) \\ &= (2a \sin\theta)^{1/2} Y_{l\mu}(\theta, \varphi)\end{aligned} \tag{2.8}$$

where $Y_{l\mu}(\theta, \varphi)$ denotes a spherical harmonic. Hence, the construction of a dynamical potential algebra for the subfamily of symmetrical Pöschl–Teller potentials is equivalent to that of a dynamical algebra for the spherical harmonics.

It is important to note that the correspondence between the wavefunctions $\psi_n^{(\mu)}(x)$ and the Legendre functions $P_l^\mu(\cos\theta)$, or between the extended wavefunctions $\Psi_n^{(\mu)}(x, \varphi)$ and the spherical harmonics $Y_{l\mu}(\theta, \varphi)$ is not one-to-one. Indeed, from (2.3), the functions $\psi_n^{(\mu)}(x)$ may only be considered as true wavefunctions for positive μ values. On the other hand, from the symmetry properties of Legendre functions (Biedenharn and Louck 1981), it results that

$$\psi_n^{(\mu)}(x) = (-1)^\mu \psi_n^{(-\mu)}(x). \tag{2.9}$$

Hence, for negative μ values, the functions $\psi_n^{(\mu)}(x)$ are but replicas of the wavefunctions. Moreover, the functions $\psi_n^{(0)}(x)$ have no counterpart in the set of wavefunctions, and must therefore be considered as unphysical. In conclusion, a relation between the (extended) solutions of the one-parameter Pöschl-Teller equation and the Legendre functions (spherical harmonics) can be obtained only at the cost of adding to the former one replica of the whole set, as well as some unphysical functions.

In the next section, we shall proceed to construct a potential algebra for the symmetrical Pöschl-Teller potentials, then extend it to a dynamical potential algebra.

3. The so(3) potential algebra of the symmetrical Pöschl-Teller potentials and its embedding into an so(3, 1) dynamical potential algebra

Equation (2.7) states that the wavefunctions $\psi_n^{(\mu)}(x)$, corresponding to the symmetrical Pöschl-Teller potentials, form the $m = 0$ subset of the set of wavefunctions associated with the two-parameter potentials. As we showed in I that so(4) and sl(4, $\mathbb{R}$) are potential and dynamical potential algebras for the two-parameter potentials, respectively, one may suspect that restriction to appropriate subalgebras will yield potential and dynamical potential algebras for the symmetrical potentials.

Let us first consider the so(4) $\simeq$ su(2) $\oplus$ su(2) potential algebra of the two-parameter Pöschl-Teller potentials, whose generators $\tilde{J}_0$, $\tilde{J}_\pm$, $\tilde{K}_0$ and $\tilde{K}_\pm$ were defined in equation (3.18) of I. Only those generators, which do not change the m value, leave invariant the subspace spanned by the wavefunctions with $m = 0$. Such operators are $\tilde{J}_0$ and $\tilde{J}_\pm$, and will henceforth be denoted by $\tilde{L}_0$ and $\tilde{L}_\pm$†. After deleting their dependence on γ and substituting φ for α, they can be written as

$$\tilde{L}_0 = -\mathrm{i}\partial_\varphi \qquad \tilde{L}_\pm = \mathrm{e}^{\pm\mathrm{i}\varphi}[\mp(2a)^{-1}\partial_x - \mathrm{i}\cot(2ax)\partial_\varphi \pm \tfrac{1}{2}\cot(2ax). \tag{3.1}$$

It can be easily checked that they satisfy both the so(3) commutation relations

$$[\tilde{L}_0, \tilde{L}_\pm] = \pm\tilde{L}_\pm \qquad [\tilde{L}_+, \tilde{L}_-] = 2\tilde{L}_0 \tag{3.2}$$

and the so(3) hermiticity properties

$$(\tilde{L}_0)^\dagger = \tilde{L}_0 \qquad (\tilde{L}_\pm)^\dagger = \tilde{L}_\mp \tag{3.3}$$

with respect to the measure $\mathrm{d}x\, \mathrm{d}\varphi$.

Their action on the extended wavefunctions (2.8) can be directly obtained from equation (3.19) of I, and is given by the relations

$$\begin{aligned} \tilde{L}_0\Psi_n^{(\mu)}(x, \varphi) &= \mu\Psi_n^{(\mu)}(x, \varphi) \\ \tilde{L}_+\Psi_n^{(\mu)}(x, \varphi) &= [n(2\mu + n + 1)]^{1/2}\Psi_{n-1}^{(\mu+1)}(x, \varphi) \end{aligned} \tag{3.4}$$

† These operators should not be confused with the so(4) generators $\tilde{L}_{\mu\nu}$ considered in I.

and a similar equation for $\tilde{L}_-$, obtained from (3.3). Moreover, the so(3) Casimir operator $\tilde{\boldsymbol{L}}^2$ is connected with the Hamiltonian of equation (2.2) (where $-\mathrm{i}\partial_\varphi$ has been substituted for μ) by the relation

$$\tilde{\boldsymbol{L}}^2 \equiv \tfrac{1}{2}(\tilde{L}_+\tilde{L}_- + \tilde{L}_-\tilde{L}_+) + \tilde{L}_0^2 = M(2\hbar^2 a^2)^{-1}H - \tfrac{1}{4}. \tag{3.5}$$

Its action on the extended wavefunctions is given by

$$\tilde{\boldsymbol{L}}^2\Psi_n^{(\mu)}(x,\varphi) = l(l+1)\Psi_n^{(\mu)}(x,\varphi) = (\Lambda_n - \tfrac{1}{4})\Psi_n^{(\mu)}(x,\varphi) \tag{3.6}$$

where l is defined in (2.6).

The extended wavefunctions $\Psi_n^{(\mu)}(x,\varphi)$, corresponding to a given l value, therefore belong to the carrier space of an so(3) irrep characterised by l. Since the so(3) generators $\tilde{L}_+$ and $\tilde{L}_-$ give rise to transitions between extended wavefunctions corresponding to the same energy, but to potential strengths differing by integers, we conclude that so(3) is a potential algebra for the symmetrical Pöschl–Teller potentials.

A similar procedure may be used to construct an so(3) symmetry algebra for the spherical harmonics, considered as special cases of rotation matrix elements. This is just the standard angular momentum algebra in spherical coordinates, which will be considered again in the next section.

Let us now turn to the sl(4, $\mathbb{R}$) dynamical potential algebra of the two-parameter Pöschl–Teller potentials, generated by $\tilde{J}_0$, $\tilde{J}_\pm$, $\tilde{K}_0$, $\tilde{K}_\pm$ and the operators $\tilde{T}_{\sigma\tau}$, σ, $\tau = +1$, 0, -1, defined in equation (4.18) of I. Those operators, which leave m unchanged, are $\tilde{J}_0$, $\tilde{J}_\pm$ and $\tilde{T}_{\sigma 0}$, $\sigma = +1, 0, -1$. After deleting their dependence on γ, substituting φ for α, and renormalising $\tilde{T}_{\sigma 0}$, we obtain the operators $\tilde{L}_0$ and $\tilde{L}_\pm$, defined in (3.1), as well as the operators

$$\tilde{A}_0 = \mathrm{i}(2a)^{-1}\sin(2ax)\partial_x + \tfrac{1}{4}(2\mathrm{i} - \eta)\cos(2ax) \tag{3.7a}$$

and

$$\tilde{A}_\pm = \mathrm{e}^{\pm\mathrm{i}\varphi}[\mathrm{i}(2a)^{-1}\cos(2ax)\partial_x \mp \operatorname{cosec}(2ax)\partial_\varphi - \tfrac{1}{2}\mathrm{i}\operatorname{cosec}(2ax) - \tfrac{1}{4}(2\mathrm{i} - \eta)\sin(2ax)] \tag{3.7b}$$

where, as in I, η is a parameter which may take any real value.

Since the operators (3.7) satisfy the commutation relations

$$\begin{array}{lll} [\tilde{L}_0, \tilde{A}_0] = 0 & [\tilde{L}_0, \tilde{A}_\pm] = \pm\tilde{A}_\pm & \\ [\tilde{L}_\pm, \tilde{A}_\pm] = 0 & [\tilde{L}_\pm, \tilde{A}_\mp] = \pm 2\tilde{A}_0 & [\tilde{L}_\pm, \tilde{A}_0] = \mp\tilde{A}_\pm \\ [\tilde{A}_0, \tilde{A}_\pm] = \mp\tilde{L}_\pm & [\tilde{A}_+, \tilde{A}_-] = -2\tilde{L}_0 & \end{array} \tag{3.8}$$

and the hermiticity properties

$$(\tilde{A}_0)^\dagger = \tilde{A}_0 \qquad (\tilde{A}_\pm)^\dagger = \tilde{A}_\mp \tag{3.9}$$

with respect to the measure $\mathrm{d}x\,\mathrm{d}\varphi$, we conclude that, together with $\tilde{L}_0$ and $\tilde{L}_\pm$, they generate an so(3, 1) algebra.

Their action on the extended wavefunctions directly results from equations (4.19) and (4.20) of I, and is given by

$$\begin{aligned} \tilde{A}_0\Psi_n^{(\mu)}(x,\varphi) = {} & -[\mathrm{i}(\mu+n+1) - \tfrac{1}{4}\eta][(n+1)(2\mu+n+1)]^{1/2} \\ & \times[(2\mu+2n+1)(2\mu+2n+3)]^{-1/2}\Psi_{n+1}^{(\mu)}(x,\varphi) + [\mathrm{i}(\mu+n) + \tfrac{1}{4}\eta] \\ & \times[n(2\mu+n)]^{1/2}[(2\mu+2n-1)(2\mu+2n+1)]^{-1/2}\Psi_{n-1}^{(\mu)}(x,\varphi) \end{aligned} \tag{3.10a}$$

$$\tilde{A}_+\Psi_n^{(\mu)}(x,\varphi)=[\mathrm{i}(\mu+n+1)-\tfrac{1}{4}\eta][(2\mu+n+1)(2\mu+n+2)]^{1/2}$$
$$\times[(2\mu+2n+1)(2\mu+2n+3)]^{-1/2}\Psi_n^{(\mu+1)}(x,\varphi)+[\mathrm{i}(\mu+n)+\tfrac{1}{4}\eta]$$
$$\times[(n-1)n]^{1/2}[(2\mu+2n-1)(2\mu+2n+1)]^{-1/2}\Psi_{n-2}^{(\mu+1)}(x,\varphi) \qquad (3.10b)$$

and a similar relation for $\tilde{A}_-$, obtained from (3.9). From (3.10*a*), it follows that the operator $\tilde{A}_0$ gives rise to transitions between extended wavefunctions $\Psi_n^{(\mu)}(x,\varphi)$ and $\Psi_{n'}^{(\mu)}(x,\varphi)$, $n'=n-1$, $n+1$, corresponding to the same potential, but to different energies. The so(3, 1) algebra is therefore a dynamical potential algebra for the symmetrical Pöschl-Teller potentials.

Energy raising and lowering operators $\mathscr{R}_n$, $\mathscr{L}_n$, corresponding to the ladder operators of the factorisation method (Infeld and Hull 1951), can be easily constructed in the enveloping algebra of so(3, 1). A possible choice is

$$\mathscr{R}_n=(\mathscr{L}_{n+1})^\dagger=(2\mu+n+1)\tilde{A}_0+\tilde{A}_-\tilde{L}_+ \qquad (3.11)$$

where

$$\mathscr{R}_n\Psi_n^{(\mu)}(x,\varphi)=[-\mathrm{i}(\mu+n+1)+\tfrac{1}{4}\eta][(n+1)(2\mu+n+1)(2\mu+2n+1)]^{1/2}$$
$$\times(2\mu+2n+3)^{-1/2}\Psi_{n+1}^{(\mu)}(x,\varphi) \qquad (3.12)$$

and

$$\mathscr{L}_n\Psi_n^{(\mu)}(x,\varphi)=[\mathrm{i}(\mu+n)+\tfrac{1}{4}\eta][n(2\mu+n)(2\mu+2n-1)]^{1/2}$$
$$\times(2\mu+2n+1)^{-1/2}\Psi_{n-1}^{(\mu)}(x,\varphi). \qquad (3.13)$$

Raising and lowering operators $\mathscr{R}$, $\mathscr{L}$, which do not refer to the index n of the function being operated upon, can be obtained from (3.11) by considering the operator

$$\tilde{L}\equiv(\tilde{\boldsymbol{L}}^2+\tfrac{1}{4})^{1/2}-\tfrac{1}{2} \qquad (3.14)$$

whose eigenvalue, corresponding to $\Psi_n^{(\mu)}$, is $l=\mu+n$. They are given by

$$\mathscr{R}=\mathscr{L}^\dagger=\tilde{A}_0(\tilde{L}+\tilde{L}_0+1)+\tilde{A}_-\tilde{L}_+ \qquad (3.15)$$

and act on $\Psi_n^{(\mu)}$ in the same way as $\mathscr{R}_n$ and $\mathscr{L}_n$, respectively.

A similar procedure may be used to construct an so(3, 1) dynamical algebra for the spherical harmonics, considered as special cases of rotation matrix elements. Such an algebra was previously obtained by another method (Vilenkin 1968).

As shown by (3.4) and (3.10), all the functions $\Psi_n^{(\mu)}(x,\varphi)$, for which $\mu=1,2,\ldots,$ and $n=0,1,2,\ldots,$ belong to the carrier space of a single degenerate continuous unitary irrep of so(3, 1) (Gel'fand *et al* 1963, Naimark 1964). As additional evidence, we note that, in the realisation defined by (3.1) and (3.7), the two so(3, 1) Casimir operators

$$\tilde{\boldsymbol{L}}^2-\tilde{\boldsymbol{A}}^2=\tfrac{1}{2}(\tilde{L}_+\tilde{L}_-+\tilde{L}_-\tilde{L}_+)+\tilde{L}_0^2-\tfrac{1}{2}(\tilde{A}_+\tilde{A}_-+\tilde{A}_-\tilde{A}_+)-\tilde{A}_0^2 \qquad (3.16a)$$

and

$$\tilde{\boldsymbol{L}}\cdot\tilde{\boldsymbol{A}}=\tfrac{1}{2}(\tilde{L}_+\tilde{A}_-+\tilde{L}_-\tilde{A}_+)+\tilde{L}_0\tilde{A}_0 \qquad (3.16b)$$

assume unique numerical values, given by

$$\tilde{\boldsymbol{L}}^2-\tilde{\boldsymbol{A}}^2=-1-(\tfrac{1}{4}\eta)^2 \qquad \tilde{\boldsymbol{L}}\cdot\tilde{\boldsymbol{A}}=0. \qquad (3.17)$$

Comparison with the eigenvalues $\omega(\omega+2)$ and 0 of the so(4) Casimir operators, corresponding to a one-row irrep, shows that the so(3, 1) irrep we have to deal with may be labelled by the (generalised) Young pattern $[\omega 0]$, where $\omega=-1+\tfrac{1}{4}\mathrm{i}\eta$. Note that a basis of its carrier space includes not only the extended wavefunctions $\Psi_n^{(\mu)}(x, \varphi)$, where $\mu = 1, 2, \ldots$, and $n = 0, 1, 2, \ldots$, but also the additional functions $\Psi_n^{(\mu)}(x, \varphi)$, where $\mu = 0, -1, -2, \ldots$, and $n = 0, 1, 2, \ldots$.

In conclusion, we have proved that the so(3) and so(3, 1) subalgebras of so(4) and sl(4, ℝ) provide us with potential and dynamical potential algebras for the symmetrical Pöschl–Teller potentials, respectively. Their construction and the determination of their action on the wavefunctions have been carried out in a straightforward way from the corresponding results for the two-parameter Pöschl–Teller potentials. In the next section, we shall contrast this approach with another one leading to an sp(4, ℝ) dynamical potential algebra.

4. The sp(4, ℝ) dynamical potential algebra of the symmetrical Pöschl–Teller potentials

By applying the algebraic version (Miller 1964, 1968, Kaufman 1966) of the factorisation method (Infeld and Hull 1951), Humi (1987) has recently constructed a dynamical algebra for the spherical harmonics, which he claimed to be so(3, 2) ≃ sp(4, ℝ). However, if its generators have appropriate commutation relations, they do not have the required hermiticity properties to qualify for a skew-Hermitian representation of so(3, 2) ≃ sp(4, ℝ) generators (corresponding to a unitary representation of the associated group). The origin of this shortcoming can be traced back to the factorisation method. The generators are indeed obtained from ladder operators arising from both factorisations of class I and II, and corresponding to different scalar products. In the present section, we shall first show how such a drawback can be cured and a skew-Hermitian representation of so(3, 2) ≃ sp(4, ℝ) constructed, and will then apply the results obtained for the spherical harmonics to the symmetrical Pöschl–Teller potentials.

The Legendre functions $P_l^{\mu}(\cos\theta)$ are solutions of the differential equation

$$[-\mathrm{d}^2_{\theta\theta} - \cot\theta\, \mathrm{d}_\theta + \mu^2 \operatorname{cosec}^2\theta - l(l+1)]P_l^{\mu}(\cos\theta) = 0. \tag{4.1}$$

Raising and lowering operators in μ are obtained from a class I factorisation (Infeld and Hull 1951). After introduction of one extra angular variable $\varphi \in [0, 2\pi)$, they lead to the standard angular momentum operators

$$L_\pm = \mathrm{e}^{\pm\mathrm{i}\varphi}(\pm\partial_\theta + \mathrm{i}\cot\theta\,\partial_\varphi) \tag{4.2}$$

acting on the spherical harmonics $Y_{l\mu}(\theta, \varphi)$ as follows:

$$L_\pm Y_{l\mu}(\theta, \varphi) = [(l \mp \mu)(l \pm \mu + 1)]^{1/2} Y_{l,\mu\pm1}(\theta, \varphi). \tag{4.3}$$

Together with

$$L_0 = -\mathrm{i}\partial_\varphi \tag{4.4}$$

characterised by the property

$$L_0 Y_{l\mu}(\theta, \varphi) = \mu Y_{l\mu}(\theta, \varphi) \tag{4.5}$$

the operators L_+ and L_- span an so(3) algebra. Their commutation properties and hermiticity properties are

$$[L_0, L_\pm] = \pm L_\pm \qquad [L_+, L_-] = 2L_0 \tag{4.6}$$

and

$$(L_0)^\dagger = L_0 \qquad (L_\pm)^\dagger = L_\mp. \tag{4.7}$$

Raising and lowering operators in l are derived from a class II factorisation (Infeld and Hull 1951). By introducing an additional dependence on one extra angular variable $\chi \in [0, 2\pi)$, they give rise to the operators (Humi 1987)

$$\bar{U}_\pm = e^{\pm i\chi}[\pm \sin\theta\, \partial_\theta + \cos\theta(-i\,\partial_\chi + \tfrac{1}{2} \pm \tfrac{1}{2})]. \tag{4.8}$$

Together with

$$U_0 = -i\partial_\chi + \tfrac{1}{2} \tag{4.9}$$

the latter span a Lie algebra since

$$[U_0, \bar{U}_\pm] = \pm \bar{U}_\pm \qquad [\bar{U}_+, \bar{U}_-] = -2U_0. \tag{4.10}$$

The action of the operators U_0 and $\bar{U}_\pm$ on the functions

$$Z_{l\mu}(\theta, \varphi, \chi) = (2\pi)^{-1/2} \exp(il\chi)\, Y_{l\mu}(\theta, \varphi) \tag{4.11}$$

is given by

$$U_0 Z_{l\mu}(\theta, \varphi, \chi) = (l + \tfrac{1}{2}) Z_{l\mu}(\theta, \varphi, \chi) \tag{4.12}$$

and

$$\bar{U}_+ Z_{l\mu}(\theta, \varphi, \chi) = [(l-\mu+1)(l+\mu+1)(2l+1)]^{1/2}(2l+3)^{-1/2} Z_{l+1,\mu}(\theta, \varphi, \chi) \tag{4.13a}$$

$$\bar{U}_- Z_{l\mu}(\theta, \varphi, \chi) = [(l-\mu)(l+\mu)(2l+1)]^{1/2}(2l-1)^{-1/2} Z_{l-1,\mu}(\theta, \varphi, \chi) \tag{4.13b}$$

respectively. Equations (4.13*a*) and (4.13*b*) directly result from the action of the ladder operators of the factorisation method on $Y_{l\mu}(\theta, \varphi)$ by taking into account the change of normalisation occurring when going from one factorisation to the other (Infeld and Hull 1951, Humi 1987).

From (4.10), Humi concludes that the operators U_0 and $\bar{U}_\pm$ (that he actually denotes by K_0 and $K_\pm$) span an so(2, 1) algebra. However, since the functions $Z_{l\mu}(\theta, \varphi, \chi)$ are orthonormal with respect to a scalar product defined in terms of the measure $\sin\theta\, d\theta\, d\varphi\, d\chi$, from (4.12) and (4.13), it follows that

$$(U_0)^\dagger = U_0 \tag{4.14}$$

but that

$$(\bar{U}_+)^\dagger \neq \bar{U}_- \tag{4.15}$$

with respect to this scalar product. This can also be shown directly on the differential operators (4.8) and (4.9). Hence, the operators U_0 and $\bar{U}_\pm$ do not provide a skew-Hermitian representation of so(2, 1).

Operators with the right Hermitian conjugation properties can be constructed in the following way. From (4.12), we note that, in the space spanned by the orthonormal functions $Z_{l\mu}(\theta, \varphi, \chi)$, $l = 0, 1, 2, \ldots, -l \leq \mu \leq l$, the Hermitian operator U_0 is positive definite. Hence, in such a space, the operators

$$U_\pm = U_0^{1/2} \bar{U}_\pm U_0^{-1/2} \tag{4.16}$$

are well defined. It can be easily shown that, together with U_0, they satisfy commutation relations similar to (4.10). Moreover, from (4.12) and (4.13), it follows that their action on the orthonormal functions $Z_{l\mu}(\theta, \varphi, \chi)$ is given by

$$U_+ Z_{l\mu}(\theta, \varphi, \chi) = [(l-\mu+1)(l+\mu+1)]^{1/2} Z_{l+1,\mu}(\theta, \varphi, \chi) \tag{4.17a}$$

and

$$U_- Z_{l\mu}(\theta, \varphi, \chi) = [(l-\mu)(l+\mu)]^{1/2} Z_{l-1,\mu}(\theta, \varphi, \chi) \tag{4.17b}$$

showing that

$$(U_+)^\dagger = U_-. \tag{4.18}$$

The operators U_0 and $U_\pm$ are therefore the searched-for so(2, 1) generators.

From (4.12) and (4.17), we conclude that the functions $Z_{l\mu}(\theta, \varphi, \chi)$, corresponding to $l = |\mu|, |\mu|+1, \ldots$, and a given μ value, span the carrier space of an so(2, 1) unitary irrep, belonging to the positive discrete series. The lowest weight function is $Z_{|\mu|,\mu}(\theta, \varphi, \chi)$ and its weight $|\mu|+\frac{1}{2}$ may be used to characterise the irrep. The so(2, 1) Casimir operator

$$\boldsymbol{U}^2 = U_+ U_- - U_0^2 + U_0 \tag{4.19}$$

is such that

$$\boldsymbol{U}^2 Z_{l\mu}(\theta, \varphi, \chi) = -(|\mu|+\tfrac{1}{2})(|\mu|-\tfrac{1}{2}) Z_{l\mu}(\theta, \varphi, \chi). \tag{4.20}$$

By forming the commutators of U_+ and U_- with L_+ and L_- (now acting in the space of functions $Z_{l\mu}$), we obtain four additional operators:

$$V_\pm = \pm[L_\pm, U_\pm] \qquad W_\pm = \pm[L_\mp, U_\pm]. \tag{4.21}$$

The latter may also be written as

$$V_\pm = U_0^{1/2} \bar{V}_\pm U_0^{-1/2} \qquad W_\pm = U_0^{1/2} \bar{W}_\pm U_0^{-1/2} \tag{4.22}$$

where

$$\begin{aligned}\bar{V}_\pm &= \pm[L_\pm, \bar{U}_\pm] \\ &= \mathrm{e}^{\pm\mathrm{i}(\varphi+\chi)}[\pm\cos\theta\, \partial_\theta + \mathrm{i}\,\mathrm{cosec}\,\theta\, \partial_\varphi - \sin\theta(-\mathrm{i}\partial_\chi + \tfrac{1}{2} \pm \tfrac{1}{2})]\end{aligned} \tag{4.23}$$

and

$$\begin{aligned}\bar{W}_\pm &= \pm[L_\mp, \bar{U}_\pm] \\ &= \mathrm{e}^{\mp\mathrm{i}(\varphi-\chi)}[\mp\cos\theta\, \partial_\theta + \mathrm{i}\,\mathrm{cosec}\,\theta\, \partial_\varphi + \sin\theta(-\mathrm{i}\,\partial_\chi + \tfrac{1}{2} \pm \tfrac{1}{2})]\end{aligned} \tag{4.24}$$

correspond to the operators R_+, $-L_-$, L_+ and $-R_-$ of Humi (1987), respectively.

As can be easily checked, the ten operators L_0, $L_\pm$, U_0, $U_\pm$, $V_\pm$ and $W_\pm$ close under commutation and satisfy the relations

$$(V_\pm)^\dagger = V_\mp \qquad (W_\pm)^\dagger = W_\mp \tag{4.25}$$

in addition to (4.7), (4.14) and (4.18). From their commutation relations and hermiticity properties, it can be recognised that they provide a skew-Hermitian representation of sp(4, $\mathbb{R}$). Their relation with the standard sp(4, $\mathbb{R}$) generators $E_{ij} = (E_{ji})^\dagger$, $D^\dagger_{ij} = D^\dagger_{ji}$, $D_{ij} = D_{ji} = (D^\dagger_{ij})^\dagger$, $i, j = 1, 2$ (Deenen and Quesne 1984) is

$$\begin{aligned}&E_{11} = L_0 + U_0 \qquad E_{22} = -L_0 + U_0 \qquad E_{12} = L_+ \qquad E_{21} = L_- \\ &D^\dagger_{11} = V_+ \qquad D^\dagger_{12} = U_+ \qquad D^\dagger_{22} = W_+ \\ &D_{11} = V_- \qquad D_{12} = U_- \qquad D_{22} = W_-.\end{aligned} \tag{4.26}$$

The non-zero commutators of such operators are given by

$$\begin{aligned}&[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{il} E_{kj} \\ &[E_{ij}, D^\dagger_{kl}] = \delta_{jk} D^\dagger_{il} + \delta_{jl} D^\dagger_{ik} \qquad [E_{ij}, D_{kl}] = -\delta_{ik} D_{jl} - \delta_{il} D_{jk} \\ &[D_{ij}, D^\dagger_{kl}] = \delta_{ik} E_{lj} + \delta_{il} E_{kj} + \delta_{jk} E_{li} + \delta_{jl} E_{ki}.\end{aligned} \tag{4.27}$$

The set of generators (4.26) can be divided into three subsets of raising, weight and lowering type as follows:

$$D^{\dagger}_{ij},\ E_{12};\ E_{ii};\ D_{ij},\ E_{21} \tag{4.28}$$

where the subsets are separated by semicolons.

The action of the sp(4, ℝ) generators on the functions $Z_{l\mu}(\theta, \varphi, \chi)$ can be easily determined from (4.3), (4.5), (4.12), (4.17) and (4.21), and is given by

$$\begin{aligned}
E_{11}Z_{l\mu}(\theta,\varphi,\chi) &= (l+\mu+\tfrac{1}{2})Z_{l\mu}(\theta,\varphi,\chi)\\
E_{22}Z_{l\mu}(\theta,\varphi,\chi) &= (l-\mu+\tfrac{1}{2})Z_{l\mu}(\theta,\varphi,\chi)\\
E_{12}Z_{l\mu}(\theta,\varphi,\chi) &= [(l-\mu)(l+\mu+1)]^{1/2}Z_{l,\mu+1}(\theta,\varphi,\chi)\\
D^{\dagger}_{11}Z_{l\mu}(\theta,\varphi,\chi) &= [(l+\mu+1)(l+\mu+2)]^{1/2}Z_{l+1,\mu+1}(\theta,\varphi,\chi)\\
D^{\dagger}_{12}Z_{l\mu}(\theta,\varphi,\chi) &= [(l-\mu+1)(l+\mu+1)]^{1/2}Z_{l+1,\mu}(\theta,\varphi,\chi)\\
D^{\dagger}_{22}Z_{l\mu}(\theta,\varphi,\chi) &= [(l-\mu+1)(l-\mu+2)]^{1/2}Z_{l+1,\mu-1}(\theta,\varphi,\chi)
\end{aligned} \tag{4.29}$$

and similar relations for E_{21}, D_{11}, D_{12} and D_{22}, resulting from their hermiticity properties. The set of functions $Z_{l\mu}(\theta, \varphi, \chi)$, $l = 0, 1, 2, \ldots$, $-l \leq \mu \leq l$, therefore span the carrier space of a single sp(4, ℝ) unitary irrep, belonging to the positive discrete series. The lowest weight function is $Z_{00}(\theta, \varphi, \chi)$. Its weight $(\frac{1}{2}, \frac{1}{2})$ may be used to characterise the irrep, which is denoted by $\langle\frac{1}{2}\frac{1}{2}\rangle$ (Deenen and Quesne 1984). In the present realisation, the two Casimir operators of sp(4, ℝ) assume unique numerical values. The quadratic one, for instance, is given by

$$\Gamma_2 \equiv \sum_{ij} (E_{ij}E_{ji} - D^{\dagger}_{ij}D_{ij}) - 3\sum_i E_{ii} = -\tfrac{5}{2}. \tag{4.30}$$

Since the sp(4, ℝ) and so(3, 2) algebras are isomorphic, we may also relate the ten operators L_0, $L_\pm$, U_0, $U_\pm$, $V_\pm$ and $W_\pm$ with so(3, 2) generators $M_{ab} = -M_{ba} = (M_{ab})^{\dagger}$, $a, b = 1, \ldots, 5$, satisfying the commutation relations

$$[M_{ab}, M_{cd}] = \mathrm{i}(g_{ac}M_{bd} - g_{ad}M_{bc} - g_{bc}M_{ad} + g_{bd}M_{ac}) \tag{4.31}$$

where the metric tensor is $g_{ab} = \mathrm{diag}(+1, +1, +1, -1, -1)$. The relations are

$$\begin{aligned}
&M_{12} = L_0 \qquad M_{23} = \tfrac{1}{2}(L_+ + L_-) \qquad M_{31} = -\tfrac{1}{2}\mathrm{i}(L_+ - L_-)\\
&M_{45} = U_0 \qquad M_{35} = \tfrac{1}{2}(U_+ + U_-) \qquad M_{34} = -\tfrac{1}{2}\mathrm{i}(U_+ - U_-)\\
&M_{14} = \tfrac{1}{4}\mathrm{i}(V_+ - V_- - W_+ + W_-) \qquad M_{24} = \tfrac{1}{4}(V_+ + V_- + W_+ + W_-)\\
&M_{15} = \tfrac{1}{4}(-V_+ - V_- + W_+ + W_-) \qquad M_{25} = \tfrac{1}{4}\mathrm{i}(V_+ - V_- + W_+ - W_-).
\end{aligned} \tag{4.32}$$

The description of the algebra in sp(4, ℝ) terms being much simpler, we shall pursue its analysis in so(3, 2) terms no further and proceed to apply our results to the symmetrical Pöschl-Teller potentials.

In the same way as we extended the spherical harmonics $Y_{l\mu}(\theta, \varphi)$ to functions $Z_{l\mu}(\theta, \varphi, \chi)$, let us go from $\Psi^{(\mu)}_n(x, \varphi)$ to the new functions

$$\begin{aligned}
\Xi^{(\mu)}_n(x,\varphi,\chi) &= (2\pi)^{-1/2}\exp[\mathrm{i}(\mu+n)\chi]\Psi^{(\mu)}_n(x,\varphi)\\
&= (2\pi)^{-1}\exp[\mathrm{i}\mu\varphi + \mathrm{i}(\mu+n)\chi]\psi^{(\mu)}_n(x).
\end{aligned} \tag{4.33}$$

From (2.6), (2.8) and (4.11), the latter can be expressed in terms of the functions $Z_{l\mu}(\theta, \varphi, \chi)$ by the relation

$$\Xi_n^{(\mu)}(x, \varphi, \chi) = (2a \sin\theta)^{1/2} Z_{l\mu}(\theta, \varphi, \chi). \tag{4.34}$$

Hence, sp(4, ℝ) generators acting in the space spanned by the functions $\Xi_n^{(\mu)}(x, \varphi, \chi)$, $\mu = 0, \pm 1, \pm 2, \ldots$, $n = 0, 1, 2, \ldots$, can be derived from the corresponding operators for the functions $Z_{l\mu}(\theta, \varphi, \chi)$ by applying a similarity transformation by $(\sin\theta)^{1/2}$, followed by the change of variable (2.4). The resulting operators are denoted by the same symbols with an extra tilde. In addition to the generators $\tilde{L}_0$ and $\tilde{L}_\pm$ of the so(3) potential algebra, already obtained in (3.1), we find the additional operators

$$\begin{aligned}
\tilde{U}_0 &= -\mathrm{i}\partial_\chi + \tfrac{1}{2}\\
\tilde{U}_\pm &= \tilde{U}_0^{1/2}\, \mathrm{e}^{\pm\mathrm{i}\chi}[\mp(2a)^{-1}\sin(2ax)\partial_x + \mathrm{i}\cos(2ax)\partial_\chi - \tfrac{1}{2}\cos(2ax)]\,\tilde{U}_0^{-1/2}\\
\tilde{V}_\pm &= \tilde{U}_0^{1/2}\, \mathrm{e}^{\pm\mathrm{i}(\varphi+\chi)}\{\pm(2a)^{-1}\cos(2ax)\partial_x + \mathrm{i}\,\mathrm{cosec}(2ax)\partial_\varphi + \mathrm{i}\sin(2ax)\partial_\chi\\
&\qquad - \tfrac{1}{2}[\sin(2ax) \pm \mathrm{cosec}(2ax)]\}\,\tilde{U}_0^{-1/2}\\
\tilde{W}_\pm &= \tilde{U}_0^{1/2}\, \mathrm{e}^{\mp\mathrm{i}(\varphi-\chi)}\{\mp(2a)^{-1}\cos(2ax)\partial_x + \mathrm{i}\,\mathrm{cosec}(2ax)\partial_\varphi - \mathrm{i}\sin(2ax)\partial_\chi\\
&\qquad + \tfrac{1}{2}[\sin(2ax) \pm \mathrm{cosec}(2ax)]\}\,\tilde{U}_0^{-1/2}.
\end{aligned} \tag{4.35}$$

They satisfy hermiticity properties similar to (4.14), (4.18) and (4.25), with respect to the measure $\mathrm{d}x\, \mathrm{d}\varphi\, \mathrm{d}\chi$. From them, sp(4, ℝ) generators in standard form, $\tilde{E}_{ij}$, $\tilde{D}_{ij}^\dagger$ and $\tilde{D}_{ij}$, can be obtained by using relations similar to (4.26).

From (2.6) and (4.29) the action of these operators on the extended wavefunctions $\Xi_n^{(\mu)}(x, \varphi, \chi)$, $\mu = 1, 2, \ldots$, $n = 0, 1, 2, \ldots$, is given by

$$\begin{aligned}
\tilde{E}_{11}\Xi_n^{(\mu)}(x, \varphi, \chi) &= (2\mu + n + \tfrac{1}{2})\Xi_n^{(\mu)}(x, \varphi, \chi)\\
\tilde{E}_{22}\Xi_n^{(\mu)}(x, \varphi, \chi) &= (n + \tfrac{1}{2})\Xi_n^{(\mu)}(x, \varphi, \chi)\\
\tilde{E}_{12}\Xi_n^{(\mu)}(x, \varphi, \chi) &= [n(2\mu + n + 1)]^{1/2}\Xi_{n-1}^{(\mu+1)}(x, \varphi, \chi)\\
\tilde{D}_{11}^\dagger\Xi_n^{(\mu)}(x, \varphi, \chi) &= [(2\mu + n + 1)(2\mu + n + 2)]^{1/2}\Xi_n^{(\mu+1)}(x, \varphi, \chi)\\
\tilde{D}_{12}^\dagger\Xi_n^{(\mu)}(x, \varphi, \chi) &= [(n + 1)(2\mu + n + 1)]^{1/2}\Xi_{n+1}^{(\mu)}(x, \varphi, \chi)\\
\tilde{D}_{22}^\dagger\Xi_n^{(\mu)}(x, \varphi, \chi) &= [(n + 1)(n + 2)]^{1/2}\Xi_{n+2}^{(\mu-1)}(x, \varphi, \chi)
\end{aligned} \tag{4.36}$$

and similar relations for $\tilde{E}_{21}$, $\tilde{D}_{11}$, $\tilde{D}_{12}$ and $\tilde{D}_{22}$. All such functions belong to the carrier space of an sp(4, ℝ) irrep $\langle\frac{1}{2}\frac{1}{2}\rangle$, whose lowest weight function is the unphysical function

$$\Xi_0^{(0)}(x, \varphi, \chi) = (2\pi)^{-1}[a\sin(2ax)]^{1/2}. \tag{4.37}$$

The generators $\tilde{D}_{12}^\dagger$ and $\tilde{D}_{12}$ give rise to transitions between functions $\Xi_n^{(\mu)}$ and $\Xi_{n+1}^{(\mu)}$ or $\Xi_{n-1}^{(\mu)}$, corresponding to the same potential, but to different energies, and are therefore energy raising and lowering operators, respectively. Hence, sp(4, ℝ) is an alternative dynamical potential algebra for the symmetrical Pöschl–Teller potentials.

5. Conclusion

In the present paper, we have proved that the so(3) potential algebra of the symmetrical Pöschl–Teller potentials can be embedded into either an so(3, 1) algebra, or an sp(4, ℝ) ≃ so(3, 2) one, and that both of them may serve equally well as a dynamical potential algebra.

However, if so(3, 1) can be obtained as a subalgebra of the sl(4, ℝ) dynamical potential algebra of the two-parameter Pöschl-Teller potentials, the same is not true for sp(4, ℝ). Comparison of (3.1) and (4.35) with (3.18) and (4.18) of I indeed shows that the embedding of sp(4, ℝ) into sl(4, ℝ), although conceivable in principle, cannot be carried out with the present realisations. Moreover, extension of the so(3, 2) algebra, isomorphic to sp(4, ℝ), to an so(4, 2) algebra for the two-parameter potentials must be ruled out for reasons detailed in I.

Hence, only so(3, 1) leads to a unified treatment of the one- and two-parameter Pöschl-Teller potentials, and, for this reason, should supersede sp(4, ℝ) as a dynamical potential algebra of the one-parameter symmetrical Pöschl-Teller potentials.

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